Proof that 1 is the only perfect square of the form $\underbrace{1111\ldots1}_{n}$

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Abstract

In the following paper, a proof founded upon number-theoretic results will be exposed for the theorem that, among the natural numbers of the form 1111...1, that is, constructed of n ones juxtaposed—for a given integer n greater than zero—, 1 is the sole element of such succession which verifies the property of being a perfect square, namely the square of another positive integer.

Theorem. *From the sequence of numbers of the form* $\underbrace{1111\ldots1}_{n}$*, where* $n \in \mathbb{N}$*, 1 is the single term whose square root is a natural number.*

Proof. Fix an $n \in \mathbb{N}$. Suppose then that $\underbrace{1111\ldots1}_{n}$ is a perfect square, therefore exists a number k such that $k \in \mathbb{N}$ and $k^2 = \underbrace{1111\ldots1}_{n}$. From this proposition follows—translating into modular arithmetic—the congruence equation written below:

$$k^2 \equiv \underbrace{1111\ldots1}_{n} \equiv 1 \pmod{10} \tag{1}$$

Which, after being solved by computational means through snippet 1, yields the following two possibilities:

$$k \equiv 1 \pmod{10} \text{ or } k \equiv 9 \pmod{10} \tag{2}$$

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This implies that the given natural k is compelled to be of the form $\overline{m1}$[1] or $\overline{m9}$, where $m \in \mathbb{N}_0$[2].

If m had a value of zero, then two possible values for k will arise, namely $k = 1$ and $k = 9$. $1^2 = 1$, which indicates that 1 is indeed a perfect square that belongs to our initial set, whereas 9, however, does not verify such property, since $9^2 = 81$, which is not of the form $\underbrace{1111\ldots1}_{n}$. It shall now be proven that k cannot adopt a value different from 1.

For an $m \neq 0$, on the other hand, this condition bifurcates once again into the two different possibilities:

- If $k = \overline{m1} = 10m + 1$, squaring k and knowing that it shall be of more than two digits, we reach the following congruence:

$$(10m + 1)^2 \equiv 11 \pmod{100} \tag{3}$$

  Which after further simplifications the following scenario can be concluded:

$$(10m + 1)^2 \equiv 100m^2 + 20m + 1 \equiv 20m + 1 \equiv 11 \pmod{100} \tag{4}$$

$$\implies 20m \equiv 10 \pmod{100} \tag{5}$$

  And since $\gcd(20, 100) = 20 \nmid 10$, the congruence has no solution[1]. Computational evaluation employing code extract 2 results in no possible value of m, as well.

- If $k = \overline{m9} = 10m + 9$, raising k to the second power and realizing, once again, that it must be of more than two digits in length, we can set up the following congruence equation:

$$(10m + 9)^2 \equiv 11 \pmod{100} \tag{6}$$

  Simplifying, the equality below is concluded:

$$(10m + 9)^2 \equiv 100m^2 + 180m + 81 \equiv 180m + 81 \equiv 11 \pmod{100} \tag{7}$$

$$\implies 180m \equiv -70 \equiv 30 \pmod{100} \tag{8}$$

  And following the same reasoning as for the case above, since $\gcd(180, 100) = 20 \nmid 30$, the equation has no solution[1]. After being analyzed thoroughly by software means—by snippet 3— it again results in no possible value for m whatsoever.

[1] The notation $\overline{mn}$ in order to represent the concatenation of the numbers m and n will be utilized from now on in this paper.

[2] Let $\mathbb{N}_0$ be defined as $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

Thus, it has been proven that such k does not exist unless the number in question has the value of 1 and hence, the original claim. ■

References

[1] G.E. Andrews. "Number Theory". In: Dover Books on Mathematics. Dover Publications, 1994, p. 60. ISBN: 9780486682525.

[2] Wolfram Research Inc. *Mathematica, Version 13.3*. Champaign, IL, 2023. URL: https://www.wolfram.com/mathematica.

Appendix

In order to find possible solutions to the congruence equations listed in this paper, the following Mathematica[2] code extracts were employed:

1.

$$k^2 \equiv \underbrace{1111\ldots1}_{n} \equiv 1 \pmod{10}$$

```
In[0]:= Solve[k^2 == 1, k, Modulus −> 10]
Out[0]:= {{k −> 1}, {k −> 9}}
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2.

$$(10m+1)^2 \equiv 11 \pmod{100}$$

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In[1]:= Solve[(10m+1)^2 == 11, m, Modulus −> 100]
Out[1]:= {}
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3.

$$(10m+9)^2 \equiv 11 \pmod{100}$$

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In[2]:= Solve[(10m+9)^2 == 11, m, Modulus −> 100]
Out[2]:= {}
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