An inductive proof that $\underbrace{6666\ldots6}_{n+1} \times \underbrace{666\ldots6}_{n}7 = \underbrace{4444\ldots4}_{n+1}\underbrace{2222\ldots2}_{n+1}$

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October 13, 2023

Abstract

In the following paper a proof will be shown—through the use of the principle of mathematical induction—for the result that, given an integer n greater than or equal to zero, the product of $6666\ldots6$ with $n+1$ sixes and $6666\ldots67$ with n sixes is equal to $4444\ldots42222\ldots2$ with $n+1$ fours and $n+1$ twos.

Theorem. *For any given n, such that $n \in \mathbb{N}_0$[1], it is verified that:*

$$\underbrace{6666\ldots6}_{n+1} \times \underbrace{666\ldots6}_{n}7 = \underbrace{4444\ldots4}_{n+1}\underbrace{2222\ldots2}_{n+1}$$

Proof. First, let the term a_n be defined as $a_n \equiv \sum_{k=0}^{n} 10^k$, which represents the integer composed of $n+1$ consecutive ones, and which computed for n between zero and five has the following appearance:

Term	**Value of n**	**Value of a_n**
a_0	0	1
a_1	1	11
a_2	2	111
a_3	3	1111
a_4	4	11111
a_5	5	111111

[1] Let $\mathbb{N}_0$ be defined as $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$

Therefore, we can rewrite the statement to be proven as the following expression in terms of a_n:

$$6a_n \times (6a_n + 1) = 4 \cdot 10^{n+1} a_n + 2a_n$$

Then, let this be our induction hypothesis, if we can prove that it holds for $n = 0$ and $n = m + 1$, we will show that it holds for all $n \in \mathbb{N}_0$, since the natural numbers are defined inductively in this manner.

- In the case $n = 0$, it is easily verifiable that our hypothesis is fulfilled:

$$6a_0 \times (6a_0 + 1) = 6 \cdot 1 \times (6 \cdot 1 + 1) = 6 \times 7 = 42 = 4 \cdot 10^1 \cdot 1 + 2 \cdot 1 = 4 \cdot 10^{0+1} a_0 + 2a_0$$

- For $n = m + 1$, first evaluate a_{m+1} and rewrite it in terms of a_m:

$$a_{m+1} = \sum_{k=0}^{m+1} 10^k = \sum_{k=0}^{m} 10^k + 10^{m+1} = a_m + 10^{m+1}$$

Now, manipulating the first member of our induction hypothesis for $n = m + 1$, we can transform it into the second:

$$\begin{aligned}
6a_{m+1} \times (6a_{m+1} + 1) &= 36{a_{m+1}}^2 + 6a_{m+1} = 36\left(a_m + 10^{m+1}\right)^2 + 6\left(a_m + 10^{m+1}\right) \\
&= 36\left({a_m}^2 + 2a_m 10^{m+1} + 10^{2m+2}\right) + 6\left(a_m + 10^{m+1}\right) \\
&= 36{a_m}^2 + 72a_m 10^{m+1} + 36 \cdot 10^{2m+2} + 6a_m + 6 \cdot 10^{m+1} \\
&= 2a_m + 2 \cdot 10^{m+1} + 4 \cdot 10^{m+2} a_m + 4a_m + 4 \cdot 10^{m+1} \\
&+ 32a_m 10^{m+1} + 36{a_m}^2 + 36 \cdot 10^{2m+2} \\
&= 2a_m + 2 \cdot 10^{m+1} + 4 \cdot 10^{m+2} a_m \\
&+ 4\left(a_m + 10^{m+1} + 8a_m 10^{m+1} + 9{a_m}^2 + 9 \cdot 10^{2m+2}\right) \\
&= 2a_m + 2 \cdot 10^{m+1} + 4 \cdot 10^{m+2} a_m \\
&+ 4\left(a_m + 10^{m+1} + 8a_m 10^{m+1} + 9{a_m}^2 + \underline{10^{2m+3} - 10^{2m+2}}\right) \\
&= \ldots 4\left(\left(a_m + 10^{m+1} + 8a_m 10^{m+1} + 9{a_m}^2 - 10^{2m+2}\right) + 10^{2m+3}\right) \\
&= \ldots 4\left(\left(\underline{10a_m + 1} + 8a_m 10^{m+1} + 9{a_m}^2 - 10^{2m+2}\right) + 10^{2m+3}\right) \\
&= \ldots 4\left(\left(\underline{a_m\left(10 + 8 \cdot 10^{m+1} + 9a_m\right)} + 1 - 10^{2m+2}\right) + 10^{2m+3}\right) \\
&= \ldots 4\left(\left(a_m\left(10 + \underline{9 \cdot 10^{m+1} - 1}\right) + 1 - 10^{2m+2}\right) + 10^{2m+3}\right) \\
&= \ldots 4\left(\left(a_m\left(\underline{9 + 9 \cdot 10^{m+1}}\right) + 1 - 10^{2m+2}\right) + 10^{2m+3}\right) \\
&= \ldots 4\left(\left(\underline{9a_m\left(1 + 10^{m+1}\right)} + 1 - 10^{2m+2}\right) + 10^{2m+3}\right) \\
&= \ldots 4\left(\left(\underline{\left(10^{m+1} - 1\right)\left(10^{m+1} + 1\right)} + 1 - 10^{2m+2}\right) + 10^{2m+3}\right)
\end{aligned}$$

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\begin{aligned}
&= \ldots 4\left(\underline{\cancel{10^{2m+2}} - \cancel{1} + \cancel{1} - \cancel{10^{2m+2}} + 10^{2m+3}}\right) \\
&= \ldots 4 \cdot 10^{2m+3} = 2a_m + 2 \cdot 10^{m+1} + 4 \cdot 10^{m+2}a_m + 4 \cdot 10^{2m+3} \\
&= 2(a_m + 10^{m+1}) + 4 \cdot 10^{m+2}(a_m + 10^{m+1}) \\
&= \boxed{4 \cdot 10^{m+2}a_{m+1} + 2a_{m+1}}
\end{aligned}
$$

And, in fact, if we insert $m+1$ in the second member of our induction hypothesis, we see that it is indeed equal to $4 \cdot 10^{m+2}a_{m+1} + 2a_{m+1}$.

Since our hypothesis is verified for $n = 0$ and for $n = m+1$, by the principle of mathematical induction[1] we can affirm that it is verified for all natural numbers including zero, and our statement is thereby proven. ■

References

[1] Mark Flanagan. 2017. URL: https://www.ucd.ie/mathstat/t4media/Induction_principle_2017_slides_web.pdf.